

# Homoclinic orbit and hidden attractor in the Lorenz-like system describing the fluid convection motion in the rotating cavity

G. A. Leonov,<sup>1</sup> N. V. Kuznetsov,<sup>1,2,\*</sup> and T. N. Mokaev<sup>1,2</sup>

<sup>1</sup>*Faculty of Mathematics and Mechanics, St. Petersburg State University, Peterhof, St. Petersburg, Russia*

<sup>2</sup>*Department of Mathematical Information Technology, University of Jyväskylä, Jyväskylä, Finland*

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In this paper a Lorenz-like system, describing the process of rotating fluid convection, is considered. The present work demonstrates numerically that this system, also like the classical Lorenz system, possesses a *homoclinic* trajectory and a chaotic *self-excited* attractor. However, for considered system, unlike the classical Lorenz one, along with self-excited attractor a *hidden* attractor can be localized. Analytical-numerical localization of hidden attractor is presented.

## I. INTRODUCTION

Consider the following physical problem: the convection of viscous incompressible fluid flow inside the ellipsoid

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 = 1, \quad a_1 > a_2 > a_3 > 0,$$

under the condition of stationary inhomogeneous external heating. It is assumed that the ellipsoid together with heat sources rotates with the constant velocity  $\Omega_0$  around its axis. The axis has a constant angle  $\alpha$  with the gravity vector  $g$ . This vector is stationary with respect to the ellipsoid motion. The value  $\Omega_0$  is assumed to be such that the centrifugal forces can be neglected in comparison with the influence of gravitational field. Consider the case when the ellipsoid rotates around the axis  $x_3$  and the vector  $g$  is placed in the plane  $x_1x_3$ . The temperature difference is generated along the axis  $a_1$  and a constant  $q_0$  is a gradient of this temperature. (Fig. 1).

Corresponding mathematical model (three-mode model of convection) was obtained by Glukhovsky and Dolzhansky [1] in the following form

$$\begin{cases} \dot{x} = Ayz + Cz - \sigma x, \\ \dot{y} = -xz + Ra - y, \\ \dot{z} = xy - z. \end{cases} \quad (1)$$

Here

$$\begin{aligned} \sigma &= \frac{\lambda}{\mu}, \quad Ta = \frac{\Omega_0^2}{\lambda^2}, \quad Ra = \frac{g\beta a_3 \hat{q}_1}{2a_1 a_2 \lambda \mu}, \\ A &= \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2} \cos^2 \alpha Ta^{-1}, \quad C = \frac{2a_1^2 a_2}{a_3(a_1^2 + a_2^2)} \sigma \sin \alpha, \\ x &= \mu^{-1} \left( \omega_3 + \frac{g\beta a_3 \cos \alpha}{2a_1 a_2 \Omega_0} q_3 \right), \quad y = \frac{g\beta a_3}{2a_1 a_2 \lambda \mu} q_1, \\ z &= \frac{g\beta a_3}{2a_1 a_2 \lambda \mu} q_2, \end{aligned}$$

and  $\lambda, \mu, \beta$  are the coefficients of viscosity, heat conduction, and volume expansion, respectively;  $q_1(t)$ ,  $q_2(t)$ , and  $q_3(t)$  are the projections of temperature gradients on the axes  $x_1$ ,  $x_2$  and  $x_3$ , respectively, in which case  $q_3(t) \equiv 0$ ;  $\omega_1(t)$ ,  $\omega_2(t)$ , and  $\omega_3(t)$  are the projections of the vectors of fluid angular velocities on the axis  $x_1$ ,  $x_2$ , and  $x_3$ , in which case

$$\omega_1 = -\frac{g\beta a_3}{2a_1 a_2 \Omega_0} \cos \alpha q_1, \quad \omega_2 = -\frac{g\beta a_3}{2a_1 a_2 \Omega_0} \cos \alpha q_2.$$

The parameters  $\sigma$ ,  $Ta$ , and  $Ra$  are the Prandtl, Taylor, and Rayleigh numbers, respectively.

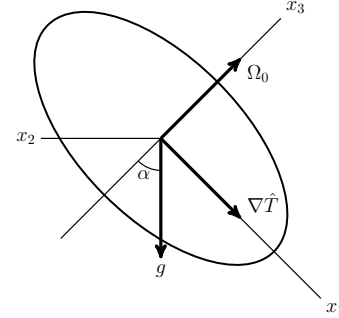


Figure 1: Illustration of the problem setting

After two sequential transformations

$$\begin{aligned} x &\rightarrow x, \quad y \rightarrow C^{-1}y, \quad z \rightarrow C^{-1}z, \\ x &\rightarrow x, \quad y \rightarrow R - \frac{\sigma}{a_0 R + 1}z, \quad z \rightarrow \frac{\sigma}{a_0 R + 1}y, \end{aligned}$$

one obtains the following system

$$\begin{cases} \dot{x} = -\sigma(x - y) - ayz \\ \dot{y} = rx - y - xz \\ \dot{z} = -z + xy, \end{cases} \quad (2)$$

where  $a_0 = A/C^2$ ,  $R = RaC$ ,

$$a = \frac{a_0 \sigma^2}{(a_0 R + 1)^2}, \quad r = \frac{R}{\sigma} (a_0 R + 1). \quad (3)$$

In the case  $a = 0$  system (2) coincides with the classical Lorenz system [2]. For the first time, system (2) with

\* Corresponding author: nkuznetsov239@gmail.com

parameters  $r, \sigma > 0$  was considered in [3, 1978]. After the linear change of variables [4] this system can be reduced to the Rabinovich system, describing the waves interaction in plasma [5–8]. As is shown in [4] system (2) describes the following physical processes: the flow of fluid convection inside the rotating ellipsoid [1], the rotation of rigid body in viscous fluid [9], the gyrostat dynamics [10, 11], the convection of a horizontal layer of fluid making the harmonic oscillations [12], and the model of Kolmogorov’s flow [13]. In [14] for system (2) in the case  $\sigma = \pm ar$  a detailed analysis of the equilibria stability and asymptotic behavior of trajectories is given and the values of parameters are obtained for which system (2) is integrable. Remark also the works [15, 16], in which the analytical and numerical study of some generalizations of system (2) and similar systems is presented. In addition, in [17] system (2) was used to describe a specific scenario of transition to chaos in low-dimensional dynamical systems — gluing bifurcations.

Note that the Glukhovsky-Dolzhansky system is sufficiently different from the classical Lorenz system. In the Lorenz system, the flow of the two-dimensional convection is considered only. In the Glukhovsky-Dolzhansky system, the flow of the three-dimensional convection is considered which can be interpreted as one of the models of ocean flows [1].

In what follows system (2) will be considered under the condition that the parameter  $a$  is positive. In this case if  $r < 1$ , then (2) has a unique equilibrium  $S_0 = (0, 0, 0)$ , which is globally asymptotically Lyapunov stable [4, 18]. If  $r > 1$ , then (2) possesses three equilibria:  $S_0 = (0, 0, 0)$  and

$$S_{1,2} = (\pm x_1, \pm y_1, z_1). \quad (4)$$

Here

$$x_1 = \frac{\sigma\sqrt{\xi}}{\sigma + a\xi}, \quad y_1 = \sqrt{\xi}, \quad z_1 = \frac{\sigma\xi}{\sigma + a\xi},$$

and the number  $\xi$  is defined as

$$\xi = \frac{\sigma}{2a^2} \left[ a(r - 2) - \sigma + \sqrt{(\sigma - ar)^2 + 4a\sigma} \right].$$

The stability of equilibria  $S_{1,2}$  depends on the parameters  $\sigma, r, a$  (see. Sec. III A).

For system (2) with the fixed  $\sigma, a$  (or with the fixed  $\sigma$  only) it is possible to observe a classical scenario of transition to the chaos similar to scenario in the Lorenz system [19]. To demonstrate this, for system (2) with the fixed parameters  $\sigma$  and  $a$  and increasing parameter  $r > 1$ , a homoclinic trajectory and a self-excited chaotic attractor are obtained numerically. Unlike the Lorenz system, for system (2) it is also possible to localize a hidden chaotic attractor.

## II. HOMOCLINIC ORBIT IN THE LORENZ-LIKE SYSTEM

Denote by  $x(t)$  a trajectory of system (2) starting at a certain initial point. In order to compute numerically a homoclinic trajectory ( $\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow -\infty} x(t) = S_0$ ), one integrates system (2) with the initial data  $x_0$  from a  $\delta$ -vicinity of the saddle point  $S_0$  and its one-dimensional unstable manifold  $W^u(S_0)$  that corresponds to a positive eigenvalue of the Jacobian matrix  $J$  at the saddle point  $S_0$ . For some parameters of system (2) this trajectory after a certain time intersects a two-dimensional plane  $M$  spanned on the eigenvectors that correspond to negative eigenvalues of  $J$ . The parameters are chosen in such a way that the point of intersection belongs to  $\delta$ -vicinity of  $S_0$ .

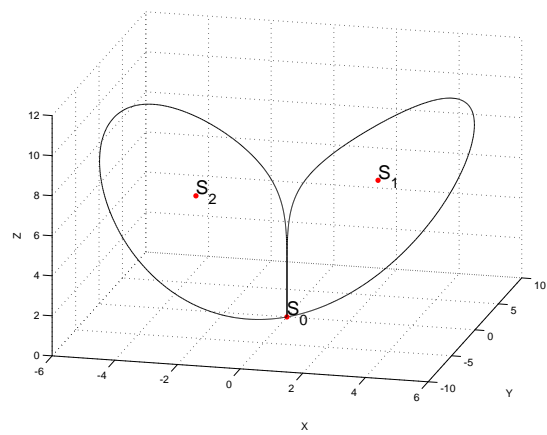


Figure 2: Approximation of the homoclinic butterfly for system (2).

Let us fix the parameters:  $\sigma = 4$  and  $a = 0.0052$  (such values were considered in [1]). For  $r = 7.44$  there is no intersection of the trajectory  $x(t)$  with the plane  $M$  (see Fig. 3a) and for  $r = 7.445$  the intersection occurs (see Fig. 3c). So, there exists an intermediate value  $r^* \in [7.44, 7.445]$  for which one can get the approximation of homoclinic orbit  $r^* = 7.4430045820796753...$  (see Fig. 3b). Note that the approximation for the symmetric homoclinic orbit can be obtained by the choose in the computational procedure the symmetric (with respect to  $S_0$ ) initial data (see Fig. 2). From an analytical point of view, the existence of homoclinic trajectory can be justified by *Fishing principle* [20–22]. The Fishing principle is based on the construction of a special two-dimensional manifold such that the separatrix of the saddle point intersects or does not intersect the manifold for two different values of a system parameter. The continuity implies the existence of some intermediate value of the parameter for which the separatrix touches the manifold. According to construction the only possibility for separatrix is to touch the saddle and thus, one can numerically localize the birth of the homoclinic orbit by changing the variable parameter.

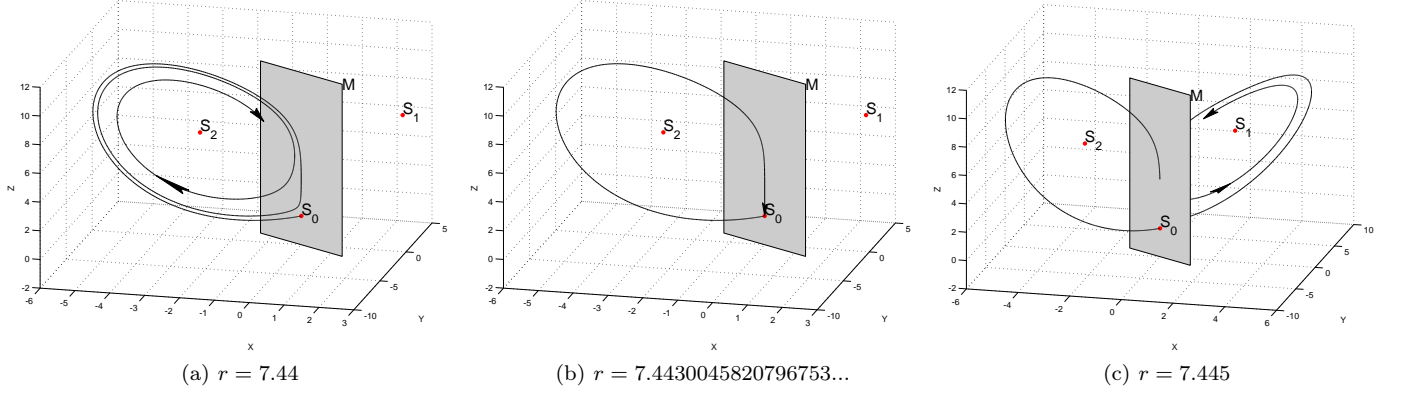


Figure 3: The birth of a homoclinic orbit in system (2) with  $\sigma = 4$ ,  $a = 0.0052$ ,  $r \in [7.44, 7.445]$ ,  $\delta = 0.01$ .

### III. CHAOTIC ATTRACTOR IN THE LORENZ-LIKE SYSTEM

#### A. Local stability analysis and computation of attractors

Let us study the stability of equilibria  $S_1, S_2$  of system (2). By the Routh-Hurwitz criterion, one can get the following

**Proposition 1 .** *If  $\sigma > 2$  and the parameters  $r$  and  $a$  satisfy the inequality*

$$p_3(\sigma, A)r^3 + p_2(\sigma, A)r^2 + p_1(\sigma, A)r + p_0(\sigma, A) < 0, \quad (5)$$

where

$$\begin{aligned} p_3(\sigma, A) &= a^2\sigma^2(\sigma - 2), \\ p_2(\sigma, A) &= -a(2\sigma^4 - 4\sigma^3 - 3a\sigma^2 + 4a\sigma + 4a), \\ p_1(\sigma, A) &= \sigma^2(\sigma^3 + 2(3a - 1)\sigma^2 - 8a\sigma + 8a), \\ p_0(\sigma, A) &= -\sigma^3(\sigma^3 + 4\sigma^2 - 16a), \end{aligned}$$

then the equilibria  $S_{1,2}$  are stable.

Let us choose the parameter  $\sigma = 4$  and, as in [1], construct the domains of stability of the equilibria  $S_{1,2}$  of system (2) in dependence on the values of parameters  $a$  and  $r$ . Then inequality (5) takes the form

$$8a^2r^3 + a(7a - 64)r^2 + (288a + 128)r + 256a - 2048 < 0. \quad (6)$$

For  $0 < a < a^*$ , where  $a^* = 0.04735\dots$ , there are three real roots:  $r_1(a) > r_2(a) > r_3(a)$ , for  $a = a^*$  two real roots:  $r_1(a)$  and  $r_2(a) = r_3(a)$ , and for  $a > a^*$  one real root:  $r_1(a)$ .

Thus, for  $0 < a < a^*$  the equilibria  $S_{1,2}$  are stable for  $r < r_3(a)$  and  $r_2(a) < r < r_1(a)$  and they are unstable in the converse case; for  $a > a^*$  the equilibria  $S_{1,2}$  are stable for  $r < r_1(a)$  (see, Fig. 4).

An oscillation in a dynamical system can be easily localized numerically if the initial conditions from its open

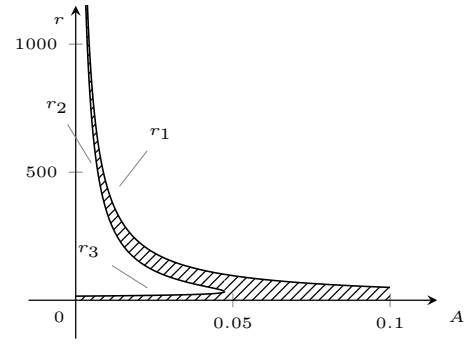


Figure 4: Domain of stability of the equilibria  $S_{1,2}$  of system (2) for  $\sigma = 4$ .

neighborhood lead to the long-time behavior that approaches the oscillation. Thus, from a computational point of view it is natural to suggest the following classification of attractors, based on the simplicity of finding the basin of attraction in the phase space:

**Definition [23–26]** An attractor is called a *hidden attractor* if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a *self-excited attractor*.

#### B. Self-excited attractor in the Lorenz-like system

For a *self-excited attractor* its basin of attraction is connected with an unstable equilibrium and, therefore, self-excited attractors can be localized numerically by the *standard computational procedure*, in which after a transient process a trajectory, started from a point of an unstable manifold in a neighborhood of an unstable equilibrium, is attracted to the state of oscillation and traces it. Thus self-excited attractors can be easily visualized.

Using the constructed domain of stability (4), one considers a qualitative behavior of trajectories of system (2) for the fixed  $\sigma = 4$ ,  $a = 0.0052$ , and  $r \in$

(16.4961242, 690.6735024). For system (2) the parameter  $r = 687.5$  is chosen.

For the above parameters the eigenvalues of equilibria of system (2) are the following

$$\begin{aligned} S_0 : & 49.9619, -1, -54.9619 \\ S_{1,2} : & 0.0295 \pm 10.729i, -6.0591 \end{aligned}$$

Thus, the equilibrium  $S_0$  is a saddle and  $S_{1,2}$  are saddle-

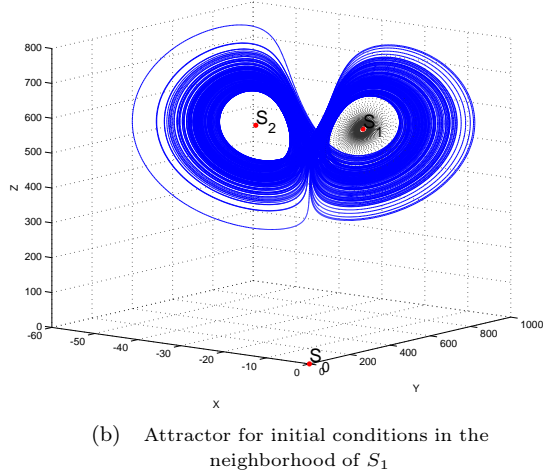
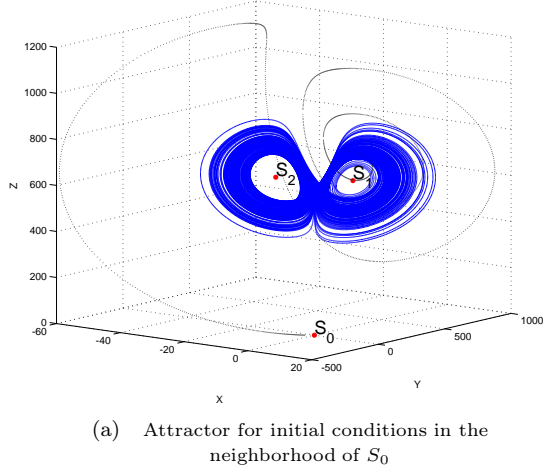


Figure 5: A self-excited attractor of system (2) for  $r = 687.5$ ,  $\sigma = 4$ ,  $a = 0.0052$ .

foci. Having taken an initial point on the unstable manifold of one of equilibria  $S_{0,1,2}$  (Fig. 5), one can easily be visualized a self-excited chaotic attractor by standard computational procedure.

For  $r \in (690.6735024, 830.4169122)$  the equilibria  $S_{1,2}$  become stable and the trajectories, starting from the neighborhood of equilibrium  $S_0$ , are attracted to  $S_1$  or  $S_2$ . The question arises whether there exists a hidden chaotic attractor in system (2) for such values of parameters? Next for the computation of hidden attractor in system (2) a special numerical procedure is considered.

### C. Hidden attractor in the Lorenz-like system

For a hidden attractor its basin of attractor is not related with unstable equilibria. The hidden attractors, for example, are the attractors in the systems with no equilibria or with only one stable equilibrium (a special case of multistable systems and coexistence of attractors). Recent examples of hidden attractors can be found in [27–38]. Multistability is often an undesired situation in many applications but the coexisting self-excited attractors can be found by the standard computational procedure. In contrast, there is no regular way to predict the existence or coexistence of hidden attractors in system. Note that one cannot guarantee the localization of attractor by the integration of trajectories with random initial data (especially for multidimensional systems) since its basin of attractor may be very small.

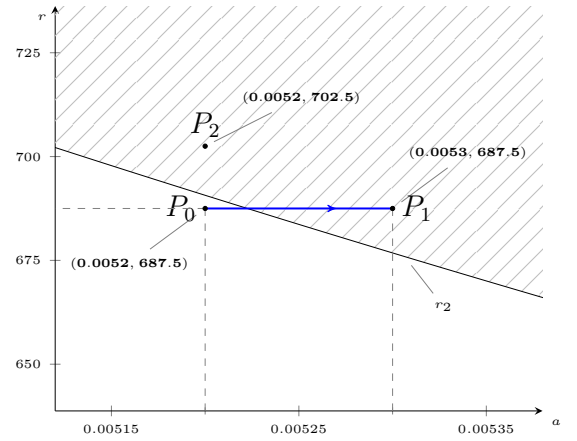


Figure 6:  $P_0$  : self-excited attractor,  $P_1$  : hidden attractor,  $P_2$  : no chaotic attractors.

One of the effective methods for numerical localization of hidden attractors in multidimensional dynamical systems is based on a *homotopy* and *numerical continuation*: it is necessary to construct a sequence of similar systems such that for the first (starting) system the initial data for numerical computation of oscillating solution (starting oscillation) can be obtained analytically, e.g, it is often possible to consider the starting system with self-excited starting oscillation. Then the transformation of this starting oscillation is tracked numerically in passing from one system to another.

Let us construct on the plane  $(a, r)$  a line segment, intersecting a boundary of the domain of stability of the equilibria  $S_1, S_2$  (see Fig. 6). Let us choose the point  $P_1$  :  $r = 687.5$ ,  $a = 0.0053$  as the finite point of the line segment. To these parameters correspond the following eigenvalues of the equilibria of system (2):

$$\begin{aligned} S_0 : & 49.9619, -1, -54.9619 \\ S_{1,2} : & -0.0968 \pm 10.4269i, -5.8063 \end{aligned}$$

It means that the equilibria  $S_{1,2}$  become stable focus-nodes.

Let us choose the point

$$P_0 : r = 687.5, a = 0.0052$$

as the initial point of the line segment. This point corresponds to the parameters for which in system (2) there exists a self-excited attractor, which can be computed by the standard procedure. Then for the considered line segment a sufficiently small partition step is chosen and a chaotic attractor in the phase of system (2) space at each iteration of the procedure is computed. The last computed point at each step is used as the initial point for the computation of the next step.

Our experiment has 8 iterations and the partition step equals  $1.25 \cdot 10^{-5}$ , respectively. At each iteration for the current trajectory that describes the attractor one computes the largest Lyapunov exponent (LLE) [39] and the Lyapunov dimension (LD) [18, 40]<sup>1</sup>.

Thus, for the selected path and selected partition it is possible to visualize a hidden attractor of system (2) (see Fig. 7).

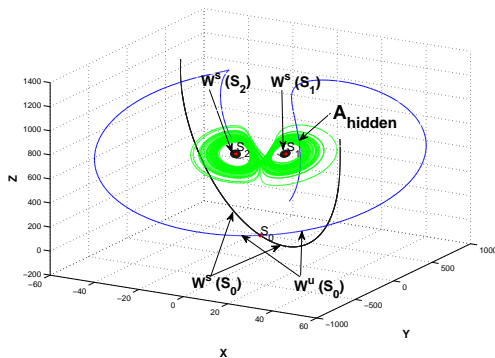


Figure 7: Hidden attractor for system (2).

Remark that hidden attractor does not exist for all points of the shaded domain in Fig. 6. E.g., there is no chaotic attractor for the point  $P_2 : r = 702.5, a = 0.0052$ .

Note also that in the work [44] the upper estimation of the Lyapunov dimension (LD) of attractor of system (2) is presented: for  $r = 687.5, a = 0.0053, \sigma = 4$  it was obtained analytically the estimation  $LD < 2.8909$  that is in a good agreement with the numerical result  $LD = 2.1293$ .

#### IV. CONCLUSIONS

In the present work by numerical methods the scenario of transition to chaos in physical model (2), describing a flow of rotating fluid convection inside the ellipsoid under horizontal heating, is demonstrated. Similarly to scenario in the classical Lorenz system, in system (2) a homoclinic trajectory and self-excited chaotic attractor are constructed. However, unlike the Lorenz system for system (2) one is able to localize numerically a hidden attractor.

#### ACKNOWLEDGMENTS

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<sup>1</sup> There are two widely used definitions of Lyapunov exponents: the upper bounds of the exponential growth rate of the norms of linearized system solutions and the upper bounds of the exponential growth rate of the singular values of linearized system fundamental matrix. While in typical case these two definitions gave the same values, for a given system they may be different and there are examples in which Benettin algorithm [41] (see, e.g., its MATLAB implementation [42]) fails to compute the correct values (see the discussion in [43])



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